Divide-and-Conquer and Searching

Divide-and-conquer In the divide-and-conquer method, we divide a problem into several subproblems (of constant fraction size), solve each subproblem recursively, and combine the solutions to the subproblems to arrive at the solution to the problem.

To be efficient, it is important to balance the sizes of the subproblems.

Searching Problems are usually stated in the form of searching for an object in a collection. In these situations, it may be useful to explicitly identify the possible scenarios. A typical algorithm for a search problem consists of a series of steps where in each step we perform some computation and ask a question to narrow down the possibilities. Each possible answer to the question reduces the problem to a subproblem where we have fewer possibilities for the object we are seeking. Moreover, the set of possible scenarios is partitioned along the possible answers to the question. For efficiency (that is, for minimizing the number of questions in the worst-case), it is important to design the algorithm so that the number of possibilities in each case is as equal as it can be to the number of possibilities in other cases.

1 Problems

Problem 1: Tromino Puzzle

Cover a $2^n \times 2^n$ $(n \geq 1)$ board missing one square with right tromnioes, which are L-shaped tiles formed by three adjacent squares. The missing square can be any of the board squares. Trominoes should cover all the squares except the missing ones with no overlaps.

A right tromino can also be viewed as a 2×2 board with exactly one missing square.

Solution: Tromino Puzzle

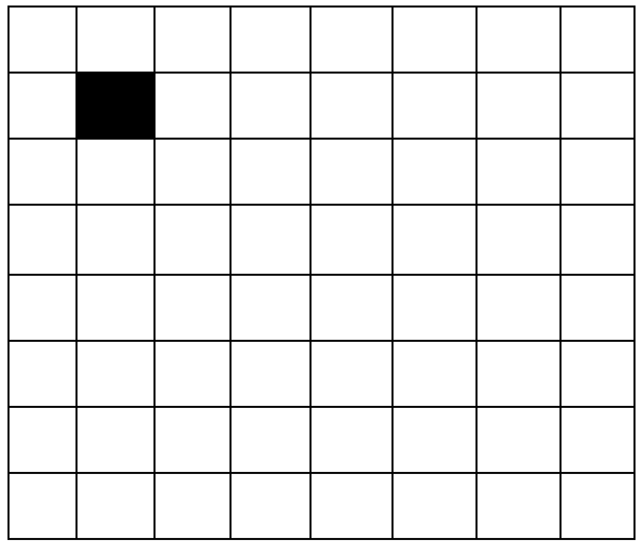
We will present an algorithm that covers a $2^n \times 2^n$ board with a missing square with L-shaped trominoes for all $n \geq 1$.

If $n = 1$, we have a 2×2 square with a missing square, which can be covered with one tromino.

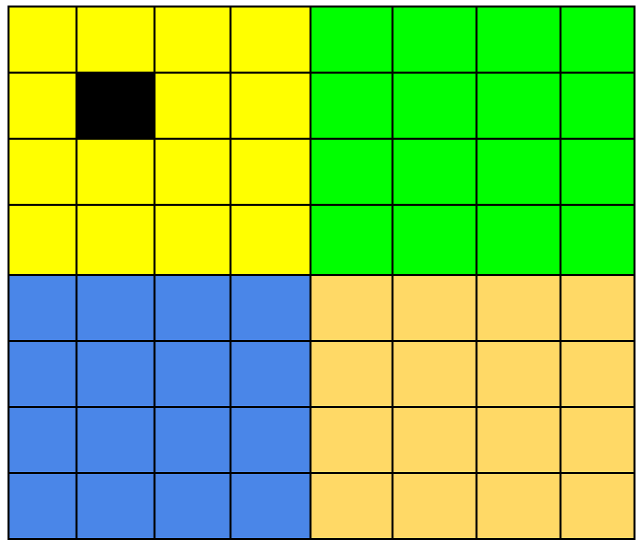
If $n \geq 2$, we divide the board into four smaller boards, each of size $2^{n-1} \times 2^{n-1}$. However, of the four smaller boards, three of them do not miss any squares. We use an appropriately oriented tromino to cover the three corner squares (which meet at the center of the board) of the three smaller boards (the ones that do not miss any squares). We now have four smaller boards, each with exactly one missing square. We cover them recursively with L-shaped trominoes.

Problem 2: A Fake among Eight Coins

There are eight identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. What is the minimum number of weighings needed to identify the fake coin with a two-pan balance scale without weights?



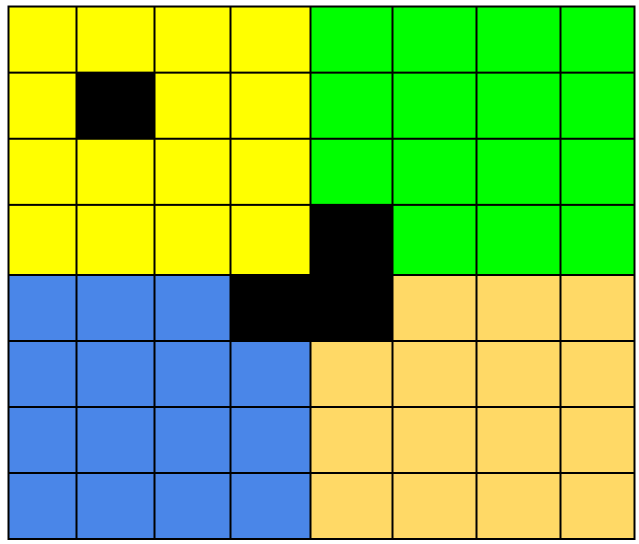
8 x 8 Board with one missing square



8 x 8 Board with one missing square

Division into 4 equal quadrants (4 x 4 boards)

The subproblems of covering the quadrants are not exactly the same type as the original problem



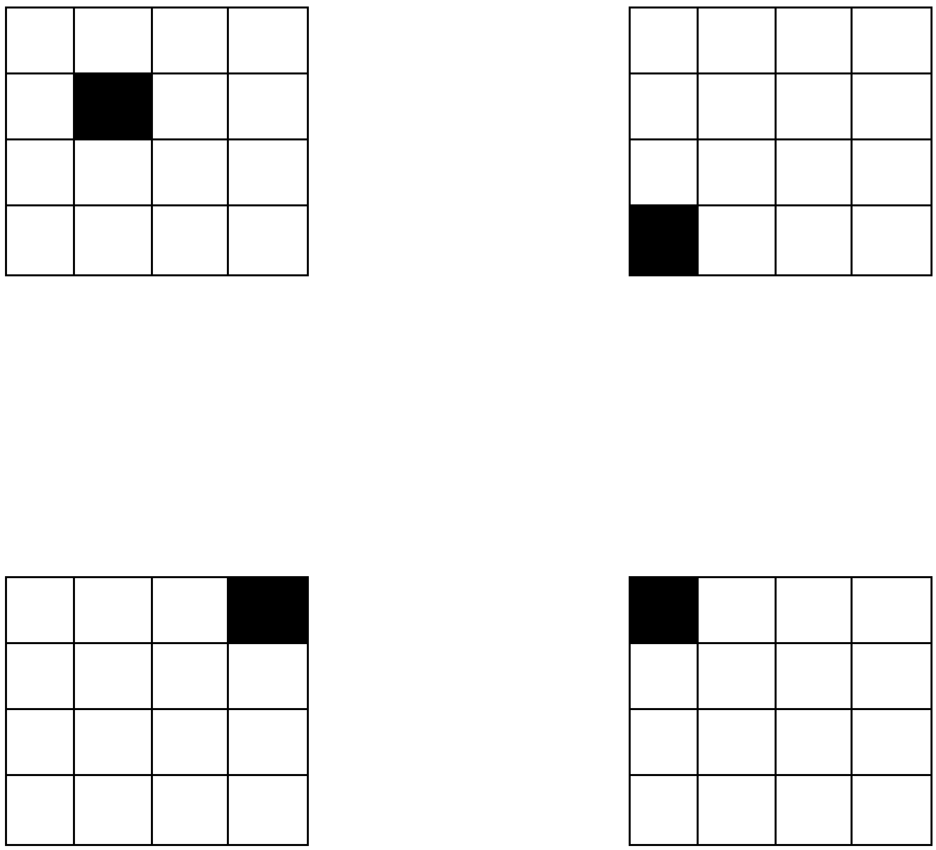
8 x 8 Board with one missing square

Division into 4 equal smaller size boards (4 x 4 boards)

The subproblems of covering the smaller boards are not exactly the same type as the original problem since three of them do not have a missing square

Use a tromino to cover the three corner squares of the three smaller boards which do not have a missing square. The three corner squares meet at the center of the board.

5



4 subproblems: 4 4x4 boards each with one missing square

Solution: A Fake among Eight Coins

Divide the coins into three groups of 3, 3, and 2 coins each and call them L, R, and E respectively. Use the two-pan balance to compare the weights of the groups L and R. If they weigh the same, the fake coin can neither be in L nor in R, so it must be in E. Take the two coins in E and compare them. The lighter coin (between the two coins in E) must be the fake coin.

If the weights of L and R are different, then the fake coin must be in the group that is lighter. Take any 2 coins from this group and weigh them. If they weigh the same then the remaining coin in the group must be fake. If one of them is lighter, the lighter coin must be the fake coin.

Problem 3: Twelve Coins

There are 12 coins identical in appearance; either all are genuine or exactly one of them is fake. It is unknown whether the fake coin is lighter or heavier than the genuine one. You have a two-pan balance scale without weights. The problem is to find whether all the coins are genuine and, if not, to find the fake coin and establish whether it is lighter or heavier than the genuine ones. Design an algorithm to solve the problem in the minimum number of weighings.

Solution: Twelve Coins

Let $c_1, c_2, \ldots, c_{12}$ denote the coins. Let $L = \{c_1, c_2, c_3, c_4\}$, $R = \{c_5, c_6, c_7, c_8\}$, and $E = \{c_9, c_{10}, c_{11}, c_{12}\}$. Compare L and R. We discuss each of the three cases below.

$L = R$

We know that all the coins in L and R are genuine. If there is a fake coin, it must be in E. Let $L' = \{c_1, c_9\}$, $R' = \{c_{10}, c_{11}\}$ and $E' = \{c_{12}\}$. Compare L' and R'.

If $L' = R'$, then the only possibilities are either all the coins are genuine or c_{12} is a fake coin (that is, it is either lighter or heavier). By comparing c_{12} with a genuine coin, we can determine whether it is genuine, lighter or heavier.

If $L' < R'$, we have three possibilities since one of c_9, c_{10}, and c_{11} must be a fake coin.

- c_9 is lighter, and c_{10}, c_{11}, and c_{12} are genuine
- c_{10} is heavier, and c_9, c_{11} and c_{12} are genuine
- c_{11} is heavier, c_9, c_{10} and c_{12} are genuine

We compare c_{10} and c_{11} to determine which one of the three alternatives holds. If c_{10} and c_{11} are equal in weight, then c_9 is lighter. Otherwise, which ever coin is heavier between them is the counterfeit and it is heavier.

If $L' > R'$, we once again have three possibilities since one of c_9, c_{10}, and c_{11} must be fake. The possibilities are c_9 is heavier, c_{10} is lighter, or c_{11} is lighter. We compare c_{10} and c_{11} to resolve this ambiguity.

$L < R$

In this case, we know that the coins in E are genuine and either one of the coins in L is lighter or one of the coins in R is heavier. Altogether we have 8 possibilities.

Let $L' = \{c_4, c_6, c_9\}$, $R' = \{c_3, c_7, c_8\}$, and $E' = \{c_1, c_2, c_5\}$. Compare L' and R'.

If $L' = R'$, then we have three possibilities: c_1 is lighter, c_2 is lighter, or c_5 is heavier. By comparing c_1 and c_2, we can resolve the uncertainty.

If $L' < R'$, we again have three possibilities: c_4 is lighter, c_7 is heavier, or c_8 is heavier. By comparing c_7 and c_8, we can find the fake coin and its relative weight.

If $L' > R'$, we have only two possibilities: c_6 is heavier or c_3 is lighter. A comparison between c_6 and c_7 should tell us which is the case.

$L > R$

This case can be handled similar to the previous case.

In conclusion, we have shown that we need three comparisons to determine whether there is a fake coin, and if so which coin it is and whether it is lighter or heavier.

Problem 4: A Stack of Fake Coins

There are 10 stacks of 10 identical-looking coins. All of the coins in one of these stacks are counterfeit, and all the coins in the other stacks are genuine. Every genuine coin weighs 10 grams, and every fake weighs 11 grams. You have an analytical scale that can determine the exact weight of any number of coins. What is the minimum number of weighings needed to identify the stack with the fake coins?

Solution: A Stack of Fake Coins

One weighing is sufficient. Let $S_1, S_2, \ldots, S_{10}$ denote the ten stacks. Create a group of coins by selecting i coins from stack S_i for every $1 \leq i \leq 10$. Use the scale to determine their weight. If the weight is w, the counterfeit stack is then S_{w-550}.

There are $\frac{(10)(10+1)}{2} = 55$ coins in the group. Each genuine coin weighs 10 grams, so the entire group, would weigh 550 grams if every S_i is a stack of genuine coins. However, there are between 1 and 10 fake coins in the group depending on which stack contains the fake coins. For any $1 \leq i \leq 12$, if S_i is the stack of fake coins, the weight of the group would be $550 + i$. Hence, subtracting 550 from the weight of the group should give us the index of the stack of fake coins.

2 Homework

Problem 5: A Fake among 33 Coins

Solve the following problems.

- There are $n = 33$ identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. What is the minimum number of weighings needed to identify the fake coin with a two-pan balance scale without weights? Describe your algorithm.
- We now consider the generalization of the previous problem for integers $n \geq 1$. There are n identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. Describe your algorithm to identify the counterfeit. How many comparisons (as a function of n) does your algorithm require in the worst case?

Solution: A Fake among 33 Coins

Divide the coins into three groups of 11 coins each. Call the groups L, R, and E. Use the two-pan balance to compare the weights of the groups L and R. If they weigh the same, the fake coin can neither be in L nor in R, so it must be in E. If the weights of L and R are different, then the fake coin must be in the group that is lighter. In all cases, we have reduced the problem to one where we have a group of 11 coins with exactly one lighter coin and we need to find the lighter coin from among 11 coins. We now proceed to the case of 11 coins.

The case of 11 coins

Divide the coins into three groups of 4, 4, and 3 coins each and proceed to compare the two groups with equal number of coins. Using the same logic as before, we arrive at one of the following two subproblems: a group of 4 coins or a group of 3 coins where we need to find the lighter coin.

The case of 4 coins

Divide the coins into three groups of 1, 1, and 2 coins each and proceed to compare the two groups with equal number of coins. We arrive at one of two subproblems: a group of 1 coin or a group of 2 coins where we need to find the lighter coin.

The case of 3 coins

Divide the coins into three groups of 1, 1, and 1 coin each and proceed to compare the two groups with equal number of coins. If the two groups have an equal weight, then the remaining coin must be the lighter coin. Otherwise, the only coin in the the lighter group between the two groups must be the lighter coin overall.

The case of 2 coins

Compare the two coins to determine the lighter coin.

The case of 1 coin

The only coin in the group must be the lighter coin.

Generalization

Suppose, instead of eight coins, there are now $n \geq 1$ coins. As before all the coins are equal in weight, except for one, which is lighter. We will argue that a general version of the previous algorithm would find the lighter coin in at most $\lceil \log_3 n \rceil$ comparisons. More precisely, we show that our algorithm performs exactly $\lceil \log_3 n \rceil$ comparisons in the worst-case.

Algorithm **SearchForLighterCoin**: If $n = 1$, the only coin must be the ligther coin, so in this case we do not need to perform any comparisons to determine the lighter coin. For the rest of the discussion, let us assume that $n \geq 2$. Without loss of generality, let n be equal to $3k$ or $3k+1$ or $3k+2$ for some $k \geq 0$. If $n = 3k$ or $3k+1$, form two groups of k coins each. Otherwise (that is, if $n = 3k+2$), form two groups of $k+1$ coins each. In either case, place the remaining coins in the third group. Call these groups L, R, and E respectively. Groups L and R will have at least one coin since $n \geq 2$.

Compare groups L and R. If they weigh the same, E has at least one coin and moreover one of the coins in it must be ligther. If the weights of L and R are different, the lighter coin must be in the lighter group. Consider the group with the lighter coin and apply the procedure recursively until the number of coins is 1.

Analysis: The analysis is a little tricky. It turns out that it is easier to work with the ternary representation of n since the divide-and-conquer scheme creates a subproblem of size about $n/3$.

Without loss of generality, let $3^k \leq n < 3^{k+1}$ for some $k \geq 0$. Write the ternary representation of n without the leading zeros. At a risk of slight ambiguity, we will use n to denote its ternary representation as well. The length of the ternary representation of n is the number of trits (for ternary digits) where we count the trits from the least significant trit to the leading trit (the leftmost non-zero trit). Observe that $\lceil \log_3 n \rceil = k + 1$ if $n > 3^k$, and $\lceil \log_3 n \rceil = k$ if $n = 3^k$. Our analysis proceeds along these two cases. In each case, we will show that the algorithm performs at most $\lceil \log_3 n \rceil$ comparisons.

We first consider the case where $n = 3^k$ and show that the algorithm performs exactly k comparisons to find the lighter coin by induction on k. If $n = 3^k$, then a comparison of groups L and R reduces the problem to that of size exactly $n/3 = 3^{k-1}$. Thus, by induction, we argue that it takes $k - 1$ more comparisons to reach the base case for a total of $k = \lceil \log_3 n \rceil$ comparisons to find the lighter coin. We do not need to perform any comparisons to figure out the lighter coin when $k = 0$ (base case for the induction).

We will now consider the case $3^k < n < 3^{k+1}$ and argue that the algorithm performs at most $k + 1$ comparisons by induction on k. If $k = 0$, then $n = 2$ in which case the algorithm performs exactly one comparison. Assume that $k \geq 1$ for the rest of the discussion.

Write the ternary representation of n as $n't$ where $n' \geq 1$ is a ternary representation of some number and t a single trit. Since $3^k < n < 3^{k+1}$, the ternary representation $n't$ is of length exactly $k + 1$. In particular, n' is of length k. After one comparison, we end up with a problem with at most $n' + 1$ coins. Note that if a number x has a ternary representation of length l, then it satisfies the following inequality: $3^l \leq x < 3^{l+1}$. Since the ternary representation n' is of length k, we get that $3^{k-1} < n' + 1 \leq 3^k$. If $n' + 1 = 3^k$, from the earlier analysis tells us that the algorithm performs exactly k more comparisons to solve the problem. Otherwise, by induction, we argue that the algorithm performs at most k more comparisons. In total, the algorithm performs at most $k + 1 = \lceil \log_3 n \rceil$ comparisons as needs to be shown.

Notes: For all $n \geq 1$, the algorithm performs exactly $\lceil \log_3 n \rceil$ comparisons in the worst case. In fact, for all $n \geq 1$, the number of comparisons in the best and the worst case differ by at most 1. To see this, observe that each comparison (except for the last comparison) could yield a subproblem of size exactly $\lceil n/3 \rceil$.

Variants

If there are two fake coins instead of one, we need 6 comparisons. Divide the 8 coins into 4 pairs and compare each pair. If any coin is lighter, there must be two coins which are lighter. These two coins are the lighter coins. If all comparisons yield equality, the fake coins must be together in the same pair. Take one coin from each of the four pairs. Use the previous algorithm to determine the lighter coin using two more comparisons.

History

References

The problems considered in this section belong to the more general class of problems, called group testing.

References

[1] D. Du and F. Hwang. *Combinatorial Group Testing and Its Applications*. Applied Mathematics. World Scientific, 2000.

[2] Piotr Indyk, Hung Q. Ngo, and Atri Rudra. Efficiently decodable non-adaptive group testing. In *Proceedings of the Twenty-first Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '10, pages 1126–1142, Philadelphia, PA, USA, 2010. Society for Industrial and Applied Mathematics.

Problem 6: Lighter or Heavier (14)

Solve the following problems.

- You have $n = 14$ identical-looking coins and a two-pan balance scale with no weights. One of the coins is a fake, but you do not know whether it is lighter or heavier than the genuine coins, which all weigh the same. Design an algorithm to determine in the minimum number of weighings whether the fake coin is lighter or heavier than the others.
- Generalize the algorithm so that it works for all $n > 2$. What is the minimum number of comparisons required?

Solution: Ligther or Heavier

A suboptimal solution: Create 3 groups of $\lfloor \frac{n}{3} \rfloor$ coins each. Call these groups A, B, and C. The remaining coins form the group E. E contains 0, 1, or 2 coins. Compare groups A and B and groups A and C. Comparison of any two groups of coins using a two-pan balance would produce one of three outcomes: the first group is lighter, the first group is heavier, or both groups have equal weight. The information obtained as a result of the two comparisons is best represented as a 3-way tree with $3 \times 3 = 9$ leaves. Each leaf corresponds to a set of possibilities. The following table captures the two-level 3-way tree. We use the notation $A < B$ to denote that the weight of coins in group A is less than the weight of the coins in group B. $A > B$ and $A = B$ are used with an analogous meaning.

A vs B	A vs C	Analysis
$A < B$	$A < C$	$B = C$. The fake coin is lighter and is in A
	$A = C$	$A = C$. The fake coin is heavier and is in B
	$A > C$	A, B, and C have distinct weights, which is impossible
$A = B$	$A < C$	$A = B$. The fake coin is heavier and is in C
	$A = C$	$A = B = C$. The fake coin is in E
	$A > C$	$A = B$. The fake coin is lighter and is in C
$A > B$	$A < C$	A, B, and C have distinct weights, which is impossible
	$A = C$	$A = C$. The fake coin is lighter and is n B
	$A > C$	$B = C$. The fake coin is heavier and is in A

For row 5, we concluded that all the coins in the groups A, B, and C are genuine and the fake coin is in the group E. To figure out whether the fake coin is lighter or heavier, take $|E|$ genuine coins from among the coins in A, B, and C. Compare this group with E to determine whether the fake coin is lighter or heavier. In total, we need a maximum of 3 comparisons.

An optimal solution: It turns out that we can devise an algorithm for this problem that uses just two comparisons.

Algorithm **DetermineLighterOrHeavier**: Let n be equal to $3k$ or $3k+1$ or $3k+2$ for some $k \geq 1$. This is without loss of generality since $n > 2$. If $n = 3k$ or $3k+1$, form two groups of k coins each. Otherwise (that is, if $n = 3k+2$), form two groups of $k+1$ coins each. In either case, place the remaining coins in the third group. Call these groups L, R, and E respectively. All groups have at least one coin since $k \geq 1$. We use the notation $|\cdot|$ to denote the size of a group. For example, $|L|$ denotes the number of coins in L.

Compare the weights of groups L and R. If the weights are equal (denoted by $L = R$), we know then that each coin in groups L and R is genuine. Select $|E|$ many coins from the groups L and R and form a new group E'. Compare the weights of the groups E and E'. If $E < E'$, then the fake coin is lighter. If $E > E'$, the fake coin is heavier. It is not possible that $E = E'$. Observe that $|L| + |R| \geq |E|$ for all $n > 2$, so it is always possible to select $|E|$ many coins from the groups L and R together.

Now consider the case $L < R$. We know that each coin in E is genuine. If $|L|$ is odd, add a genuine coin from E to L. Without loss of generality, assume that $|L| \geq 2$ is even. Split L into two subgroups of equal size. Call the subgroups L' and L''. Compare L' and L''. If $L' = L''$, we infer that the fake coin is heavier since all the coins in L are genuine. If $L' < L''$ or $L' > L''$, we infer that the fake coin is lighter. since otherwise we will have

The case $L > R$ can be handled similarly.

Thus, in call cases, we have shown that we need only two comparisons to determine whether the fake coin is lighter or heavier.

Problem 7: Max-Min Weights

Given $n > 1$ items and a two-pan balance scale with no weights, determine the lightest and the heaviest items in $\lceil 3n/2 \rceil - 2$ weighings.

Solution: Max-Min Weights

There are many different solutions to this problem, each achieving the required bound on the number of comparisons.

Solution 1: Divide the objects into pairs. Set aside the unpaired object, if it exists. Compare the objects in each pair. Add the lighter object to group A and the heavier object to group B. If there is a tie, add the one of the objects to group A and the other to group B. Add the unpaired object to both groups.

Observe that group A contains the lightest object and group B contains the heaviest object. Use the following procedure to find the lightest object from A.

Assume that the objects in A are arranged in a sequence. Scan the objects in A in order while comparing the next object to the lightest object found so far. Initially, the lightest object found so far is the first object in the sequence. After all objects in A are scanned, the lightest object found so far must be the lightest object in A.

We find the heaviest object in B by using a similar procedure.

There are $\lfloor \frac{n}{2} \rfloor$ comparisons in the first phase to determine the groups A and B. Finding the lightest object from group A requires exactly $\lceil \frac{n}{2} \rceil - 1$ comparisons. Similarly, it takes $\lceil \frac{n}{2} \rceil - 1$ comparison to find the heaviest object from group B. In total, we need exactly $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil - 1 + \lceil \frac{n}{2} \rceil - 1 = n + \lceil \frac{n}{2} \rceil - 2 = \lceil \frac{3n}{2} \rceil - 2$ comparisons.

Solution 2:

Divide the objects into two groups of $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ objects each. Call these two groups L and R respectively. For each group determine the maximum and the minimum objects. Let $\max_L$ and $\min_L$ denote the maximum and the minimum objects in group L respectively. Similarly, let $\max_R$ and $\min_R$ denote the maximum and the minimum objects in group R respectively. Compare $\max_L$ and $\max_R$ to arrive at the maximum weight object among all the objects. Compare $\min_L$ and $\min_R$ to arrive at the minimum weight object among all the objects.

It turns out that this divide-and-conquer algorithm requies exactly $\lceil 3n/2 \rceil - 2$ comparisons. The analysis is somewhat involved and will be presented elsewhere.

3 Advanced Problems

Problem 8: Tromino Tilings

For each of the three cases, prove or disprove that for every $n > 0$ all the boards of the following can be tiled by right trominoes.

A *tiling* is a cover of the of board with no overlaps.

1. $3^n \times 3^n$
2. $5^n \times 5^n$
3. $6^n \times 6^n$

Problem 9: Pancake Sorting

There are n pancakes, all of different sizes, that are stacked on top of each other. You are allowed to slip a spatula under one of the pancakes and flip over the whole stack above the spatula. The objective is to arrange the pancakes according to their size with the biggest at the bottom. Design an algorithm for solving this puzzle and determine the number of flips made by the algorithm in the worst case.

Problem 10: A Fake among n Coins: Analysis

There are $n \geq 1$ identical-looking coins; one of these coins is counterfeit and is known to be lighter than the genuine coins. Describe your algorithm to identify the counterfeit. How many comparisons (as a function of n) does your algorithm require in the worst case? Analyze your algorithm to determine the worst-case number of comparisons.

4 Challenge Problem

Problem 11: The Josephus Problem

We have n people numbered (clockwise) 1 to n around a circle. We eliminate every second person until only one survives. Determine the survivor's number.