The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

MATHEMATICAL OLYMPIADS LECTURE NOTES

The *Floor* or *Integer Part* function

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Definition

Every real number x can be written in *exactly* one way as

$$x = n + z,$$

where $n \in \mathbb{Z}$ and $0 \le z < 1$. We call n the *integer part* or *floor* of x and denote it by $[n]$ or $\lfloor n \rfloor$; and z is called the *fractional part* of x and is denoted by $\{x\}$. Thus

$$\text{for } x \in \mathbb{R}, \quad \lfloor x \rfloor \text{ is the greatest integer not exceeding } x.$$

The fractional part of x is commonly thought of as *the part after the decimal point*, but this notion is only correct for *positive* x. We define the *fractional part* by

$$\text{for } x \in \mathbb{R}, \quad \{x\} = x - \lfloor x \rfloor .$$

The notation $\lfloor x \rfloor$ and term *floor* are supposed to emphasise that $\lfloor x \rfloor \le x$. In fact, the term *integer part* leaves a sense of ambiguity when applied to negative real numbers – while the term *floor* does not. There is also a *ceiling* function: for $x \in \mathbb{R}$ the *ceiling* of x, denoted by $\lceil x \rceil$ is defined by

$$\lceil x \rceil \text{ is the least integer not less than } x.$$

Hence

$$\lceil x \rceil = \begin{cases} x = \lfloor x \rfloor & \text{if } x \in \mathbb{Z} \\ \lfloor x \rfloor + 1 & \text{if } x \notin \mathbb{Z}. \end{cases}$$

The notation $\lceil x \rceil$ and term *ceiling* are supposed to emphasise that $\lceil x \rceil \ge x$.

Properties

For $\boxed{x, y \in \mathbb{R}}$ we have the following properties:

1. $\boxed{x = \lfloor x \rfloor + \{x\}}$

2. $\boxed{x = \lfloor x \rfloor \iff x \in \mathbb{Z}}$

3. $\boxed{x = \{x\} \iff 0 \le x < 1}$

4. $\boxed{x - 1 < \lfloor x \rfloor \le x}$

5. If $k \in \mathbb{Z}$ then
   $\boxed{\lfloor x + k \rfloor = \lfloor x \rfloor + k}$
   $\boxed{\{x + k\} = \{x\}}$

6. $\boxed{\lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor \text{ is 0 or 1}}$ and hence
   $\boxed{\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor}$
   $\boxed{\{x + y\} \leq \{x\} + \{y\}}$
   since ...

   By Property 4.,

$$\begin{aligned} x + y - 1 &< \lfloor x + y \rfloor \leq x + y \\ x - 1 &< \lfloor x \rfloor \leq x, \quad (\text{i.e. } -x \leq -\lfloor x \rfloor < -x + 1) \\ y - 1 &< \lfloor y \rfloor \leq y, \quad (\text{i.e. } -y \leq -\lfloor y \rfloor < -y + 1) \\ \text{So} \quad -1 &< \lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor < 2. \end{aligned}$$

   But $\lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor \in \mathbb{Z}$.
   So $\lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor$ is 0 or 1. Hence $\lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor \geq 0$ i.e.

$$\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor .$$

   Applying Property 1 to the last inequality gives

$$\{x + y\} \leq \{x\} + \{y\} .$$

Additional properties

Suppose $\boxed{0 < \alpha \in \mathbb{R}}$ and $\boxed{n \in \mathbb{N}}$. Then the *floor* function has the following additional properties.

7. If $\alpha > 0$ and $n \in \mathbb{N}$ then $\left\lfloor \dfrac{n}{\alpha} \right\rfloor$ is the number of positive integer multiples of α not exceeding n,

   since ...

   $\alpha > 0$. So for some $\ell \in \mathbb{N}$, each of

$$\alpha, 2\alpha, \dots, \ell\alpha$$

   is less than or equal to n, and each of

$$(\ell + 1)\alpha, (\ell + 2)\alpha, \dots$$

   is greater than n. That is,

$$\begin{aligned} \ell\alpha \leq n &< (\ell + 1)\alpha \\ \text{So} \quad \ell \leq \frac{n}{\alpha} &< \ell + 1, \quad (\text{since } \alpha > 0). \end{aligned}$$

   Hence $\ell = \left\lfloor \dfrac{n}{\alpha} \right\rfloor$ is the number of positive integer multiples of α not exceeding n.