Study Guide for Predicate Logic

Aug. 28, 2015

Predicate logic improves on Propositional logic by allowing the use of *quantifiers*: for all and there exists. This introduction of quantifiers enables reasoning about groups and quantities at once.

Predicate logic requires predicates. A predicate is a statement from which some or all of the nouns have been removed. For example:

John is a student at UC San Diego

is a statement. By removing two of the nouns, we can create a predicate:

$P(x, y) = x$ is a student at y.

The *truth set* of a predicate with domain D is the subset of D where the predicate is true.

Universal quantifiers

The universal quantifer is $\forall$, read "for all". A universally quantified statement:

$\forall x \in D : P(x)$

is true if the truth set of $P(x)$ is D (that is if it is true for all elements of D).

The existential quantifer is $\exists$, read "there exists". An existentially quantified statement:

$\exists x \in D : P(x)$

is true if the truth set of $P(x)$ is not empty (that is if it is true for at least one element of D).

Negation

If an existential statement is negated, a restatement can be made with a non-negated universal statement (not unlike De Morgan's law).

For example:

$\neg \exists x \in D, P(x)$

is equivalent to:

$\forall x \in D, \neg P(x)$

Multiple quantifiers

If a quantified statement has multiple quantifiers, they are evaluated left-to-right. Consider the following statement:

$\forall x \in D, y \in E, \exists z \in F, w \in G, P(x, y, w, z)$

This states that for every single x and y, there exists a z and w where the predicate is true.

Multiple quantifiers can be considered an adversarial game. The adversary gets to choose a value for any universally quantified variables, while you get to choose a value for any existentially quantified variables. In this example, the adversary would get to choose a x and y as they see fit. In response, you'd get to choose z and w (based on the already-chosen values of x and y).

Consider the difference between

$\exists y \in D, \forall x \in E, f(x, y)$

and:

$\forall x \in E, \exists y \in D, f(x, y)$

In the former case, you must come up with a y that works no matter what x the adversary chooses. In the latter case, since the adversary chooses first, you can craft a (possibly different) y for each x the adversary chooses.