CSE 20: Midterm Solution

August 21, 2015

Name: ______________________________

Student ID:_________________________

No books and no calculators are allowed. One double-sized page of handwritten notes is allowed. If you need to make an assumption to solve a problem, state the assumption. In order to receive partial credit, you need to show your work.

Problem	Score
1	/15
2	/12
3	/10
4	/20
5	/24
6	/18
Extra Credit	/20
Total	/100

1. *15 pts.*

   (a) What is the smallest negative 9-bit twos-complement number?

   Solution: The smallest negative 9-bit twos-complement number is $-2^8 = -256$, which is 100000000_2.

   (b) The largest positive one?

   Solution: The largest positive twos-complement number is $2^8 - 1 = 255$, which is 011111111_2.

   (c) How many total distinct 9-bit twos-complement numbers are there?

   Solution: Since each 9-bits represents a distinct 9-bit twos-complement number, there are $2^9 = 512$ distinct 9-bit twos-complement numbers. An alternative count is to see that there are 256 negative numbers, 1 zero, and 255 positive numbers, for a total of 512 numbers.

2. *12 pts.*

   Convert the octal number 33653337357_8 to hexadecimal:

   Solution: Convert to binary, rearrange in groups of 4, and then convert back to octal

$$\begin{aligned}33653337357_8 &= 11\ 011\ 110\ 101\ 011\ 011\ 011\ 111\ 011\ 101\ 111_2 \\ &= 1101\ 1110\ 1010\ 1101\ 1011\ 1110\ 1110\ 1111_2 \\ &= DEADBEEF_16\end{aligned}$$

3. *10 pts.* Compute the value of:

$$\sum_{i=0}^{4} 30(i+1)^2$$

Solution:

$$\begin{aligned}\sum_{i=0}^{4} 30(i+1)^2 &= 30\sum_{i=0}^{4}(i+1)^2\\ &= 30\sum_{k=1} 5i^2\\ &= 30(1+4+9+16+25)\\ &= 30(55)\\ &= 1650\end{aligned}$$

4. *20 pts.* You are given the following truth table for S.

p	q	r	S
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

(a) Design a boolean function (using *only* the operators $\wedge$, and $\sim$) equal to S.

Solution: We can use DNF form by creating a conjunct for each row where S is one:

$$S = (\sim p \wedge q \wedge r) \vee (p \wedge \sim q \wedge \sim r).$$

Then, we can convert the $\wedge$ operators to $\vee$ using De'Morgan's Law:

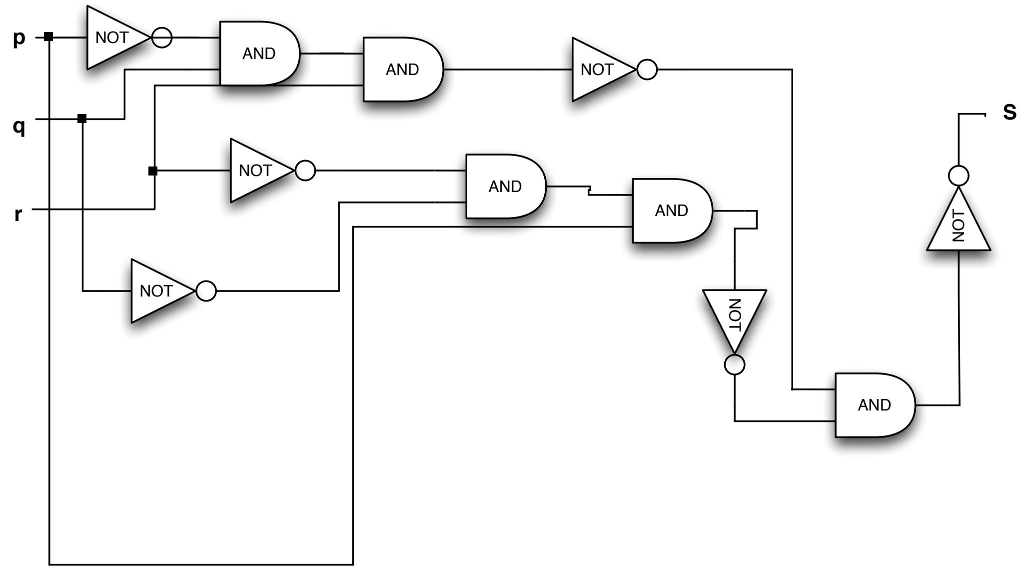
$$S = \sim (\sim (\sim p \wedge q \wedge r) \wedge \sim (p \wedge \sim q \wedge \sim r)).$$

Points:

Large errors: -3 (examples: Misapply distribute or De-Morgan's)
Medium errors: -2 (example: boxing wrong answer)
Minor errors: -1 (examples:missing $\wedge$)

(b) Draw a circuit for S using only *not* gates and *and* gates (make your *and* gates take exactly two inputs).

Solution: Use the function from part a):



Points:

Large errors: -3 (examples: 2- or 3-input not)
Medium errors: -2 (example: 3-input and, missing gate)
Minor errors: -1 (examples:missing ∼)

If the circuit didn't bear a resemblance to the original function, or if it wasn't a valid circuit, no or few points were given.

In addition, if a new function was derived (rather than using the result from part a), grading from part a) also applied (e.g., misapplication of De-Morgan's).

5. *24 pts.* Circle the true statements:

(a) $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+$ such that $xy = 1$.

Solution: True. This just says that every positive real has a multiplicative inverse. All reals other than zero have a multiplicative inverse.

(b) $\exists y \in \mathbb{R}^+$ such that $\forall x \in \mathbb{R}^+, xy = 1$.

Solution: False. This says there's a single real which is the multiplicative inverse of all positive reals.

(c) $\exists u \in \mathbb{R}^+$, such that $\forall v \in \mathbb{R}^+$, $uv < v$.

Solution: True. For example, let u be $1/2$. Then $v/2 < v$ for all positive reals v.

(d) $\forall x \in \mathbb{Z}^+$ and $\forall y \in \mathbb{Z}^+, \exists z \in \mathbb{Z}^+$ such that $z = x - y$.

Solution: False. A counter-example is $x = 3, y = 5$: the difference is -2, which is not an element of $\mathbb{Z}^+$.

(e) $\forall x \in D, (P(x) \vee Q(x))$
always has the same truth value as
$(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$.

Solution: False. Let D be the people in CSE 20, $P(x)$ be the predicate "x is male", and $Q(x)$ be the predicate "x is female". The first quantified statement is true, because everyone in class is either male or female, but the second statement is false, because it is not the case that all students in class are male, nor is it the case that all students in class are female.

Thus, the two quantified statements don't always have the same truth value.

(f) $\forall x \in D, (P(x) \wedge Q(x))$
always has the same truth value as
$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$.

Solution: True. The first statement says that every x in D is in the truth set of both $P(x)$ and $Q(x)$. The second statement says that every $x \in D$ is in the truth set of $P(x)$ and that every $x \in D$ is in the truth set of $Q(x)$. The two statements thus always have the same truth value: true if every element of D is in both truth sets, false otherwise.

6. *19 pts.* Prove directly from the definition of divisibility:

$$\forall a, b, c \in \mathbb{Z}, \text{ if } a|b \text{ and } b|c \text{ then } a|(b-c).$$

Solution: Let a, b, and c be arbitrary integers. Since $a|b$, by the definition of divisibility, there exists an integer k where $ak = b$. Similarly, since $b|c$, there exists an integer j where $bj = c$. We can rewrite $b - c$ as $ak - akj$. Factoring, we have $b - c = a(k - kj)$. Since k and j are integers and integers are closed under multiplication and subtraction, $k - kj$ is an integer. But, since $b - c$ is a times an integer, by the definition of divisibility, $a|(b-c)$.

7. *20 pts.* Extra Credit . Prove that, for all real numbers x, if $x - \lfloor x \rfloor \geq 1/2$, then $\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$.

Solution: Assume that, for all real numbers $x, x - \lfloor x \rfloor \geq 1/2$. Let x be an arbitrary real where $x - \lfloor x \rfloor \geq 1/2$. By the definition of floor, $\lfloor x \rfloor$ is the unique integer n such that $n \leq x < n+1$. Since $x < n+1$, we know that $x - n < 1$. We also know that $x - n \geq 1/2$. Multiplying by 2, we see that $1 \leq 2(x-n) < 2$.

We'll prove the equality by starting with the left hand side:

$$
\begin{aligned}
\lfloor 2x \rfloor &= \lfloor 2(n + x - n) \rfloor \\
&= \lfloor 2n + 2(x-n) \rfloor && \text{Distributive} \\
&= 2n + \lfloor 2(x-n) \rfloor && \text{Theorem 4.5.1} \\
&= 2n + 1 && \text{since } 1 \leq 2(x-n) < 2 \\
&= 2\lfloor x \rfloor + 1 && \text{by definition of } n
\end{aligned}
$$

So, we've shown that, for any real number x where $x - \lfloor x \rfloor \geq 1/2$, that $\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$.

QED.